Exercise 42

If $g(x) = x/e^x$, find $g^{(n)}(x)$.

Solution

Evaluate the first derivative using the quotient rule.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{x}{e^x}\right) \\ &= \frac{\left[\frac{d}{dx}(x)\right](e^x) - \left[\frac{d}{dx}(e^x)\right](x)}{e^{2x}} \\ &= \frac{(1)(e^x) - (e^x)(x)}{e^{2x}} \\ &= \frac{1-x}{e^x} \end{aligned}$$

Evaluate the second derivative using the quotient rule again.

$$\begin{aligned} g''(x) &= \frac{d}{dx}[g'(x)] \\ &= \frac{d}{dx}\left(\frac{1-x}{e^x}\right) \\ &= \frac{\left[\frac{d}{dx}(1-x)\right](e^x) - \left[\frac{d}{dx}(e^x)\right](1-x)}{e^{2x}} \\ &= \frac{(-1)(e^x) - (e^x)(1-x)}{e^{2x}} \\ &= -\frac{2-x}{e^x} \end{aligned}$$

Evaluate the third derivative using the quotient rule again.

$$\begin{aligned} g'''(x) &= \frac{d}{dx}[g''(x)] \\ &= \frac{d}{dx}\left(-\frac{2-x}{e^x}\right) \\ &= -\frac{\left[\frac{d}{dx}(2-x)\right](e^x) - \left[\frac{d}{dx}(e^x)\right](2-x)}{e^{2x}} \\ &= -\frac{(-1)(e^x) - (e^x)(2-x)}{e^{2x}} \\ &= \frac{3-x}{e^x} \end{aligned}$$

Therefore, recognizing the pattern,

$$g^{(n)}(x) = (-1)^{n+1}\frac{n-x}{e^x}.$$